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4.5 Case-based Reasoning

Many-valued similarity can be used to model Case-based Reasoning, too. Having n model cases, we want to consider a new one similar enough to be compared with the cases in the database. We illustrate the idea by an example, which is again from medicine.

Maximal heartbeat rate, HRmax, is a good measure of cardiorespritory fitness, indeed, population studies have shown that low fit individuals have a significantly greater risk of all-cause mortality compared with high fit subjects, indicating health-related validity of this measure. Direct measurements of HRmax, however, require expensive apparatus and laboratory facilities and the test protocols require the individual to exercise to the exhaustion. Sub-maximal test protocols provide inexpensive, safe and feasible way of testing healthy adults. Several HRmax prediction methods, based on sub-maximal exercise tests are already available. The prediction accuracy and their ability to classify fitness are acceptable on group level, but not so for individuals.

At The Tampere Research Center of Sports Medicine the following preliminary research was carried out. 57 healthy females of age 34 to 51 years got through a 2 kilometers walk test including a test for sub-maximal heartbeat rate HRsub and another test for HRmax, and resulting the following information

	age	W-ind	mlk	HR_{sub}	ww%	time	weight	length	HR_{max}
C1	34	23.39	36.4	174	75.2	15.7	68.4	171	197
C2	34	25.16	33.5	165	73.8	16.8	72.7	170	192
C3	35	21.77	38.4	150	68.7	16.1	58.9	165	192
⋮									
C57	51	18.48	39.7	140	69.0	15.7	53.4	170	173

The range of each variable was calculated first, for e.g. Body Mass Index (W-ind) it was from 18.48 to 32.66, and on this bases eight scaled fuzzy sets were greaten. Each of them generated a fuzzy similarity relation, and an experienced medical doctor weighted each factor by weights 5, 7, 4, 20, 50, 7, 5 and 2, respectively. This yield 57 total similarity relations corresponding to each case. The relevance of such treatment was tested by calculating the internal dependence of the cases. It turned out that whenever total fuzzy similarity of two different cases was greater than 0.925, then the corresponding difference in HRmax values was less or equal to 10 beats/min. Such a result is superior to other methods. The database can be used to predict HRmax for objects that go through a 2 kilometers walk test.

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In a similar manner we construct the fuzzy sets

	`Huge Area`
`High Population`	`High Amount of Births`
`High Infant Mortality`	`Long Life Expectancy`
`Many Old Aged`	`High Literacy`

corresponding to the columns 2-8. For the last two columns it is more reasonable to define a fuzzy set `Telephone Per Person` by

$$\mu_{\texttt{Telephone Per Person}}(\text{state x}) = \frac{1}{\text{1 telephone per x persons in state x}}$$

and similarly a fuzzy set `Car Per Person`. This results

State	GDT	area	pop	bir	mor	life	65	lite	tel	car
Finland	0.82	0.27	0	0.09	0	0.97	0.79	1	0.56	0.37
Denmark	1	0.01	0	0.12	0	0.97	0.86	1	0.63	0.32
Belgium	0.89	0	0.1	0.12	0.01	0.97	0.93	0.99	0.45	0.42
France	0.86	0.46	1	0.09	0.01	1	0.93	0.99	0.56	0.42
Italy	0.86	0.24	0.99	0.06	0.02	1	1	0.96	0.43	0.53
Spain	0.65	0.43	0.64	0.06	0.01	1	0.93	0.96	0.38	0.36
Slovakia	0.31	0.02	0.01	0.15	0.06	0.83	0.57	1	0.21	0.19
Bulgaria	0.20	0.07	0.07	0	0.10	0.77	0.93	0.97	0.30	0.19
Romania	0.18	0.19	0.31	0.06	0.18	0.74	0.71	0.96	0.13	0.09
Colombia	0.22	1	0.61	0.39	0.20	0.86	0.14	0.88	0.10	0.03
Tanzania	0	0.82	0.46	1	1	0	0	0.56	0	0
Nepal	0.02	0.11	0.33	0.88	0.72	0.40	0	0	0	0

Example 1 Clearly, Romania, Bulgaria and Slovakia are countries mutually equal with respect to the above information. By maximal fuzzy similarities generated by the above fuzzy sets we calculate Similar⟨Slovakia,Bulgaria⟩ = 0.9050, Similar⟨Slovakia,Romania⟩ = 0.8380 and Similar⟨Romania,Bulgaria⟩ = 0.8420.

Example 2 Typical features of an *undeveloped country* are low GDP per capita, high rate of birth, high infant mortality, short life expectancy at birth and low literacy percentage. In the light of above facts, which are the three most undeveloped countries? It is reasonable to use the following fuzzy sets

`Low GDP = (High GDP)*`
`High Amount of Births`
`High Infant Mortality`
`Short Life Expectancy = (Long Life Expectancy)*`
`Low Literacy =(High Literacy)*`

and to assume that a 'typical undeveloped country' belongs to each fuzzy set at the degree 1. Comparing now each state with such a typical country results the following degrees of total fuzzy similarity: Finland 0.06, Denmark 0.03, Belgium 0.06, France 0.05, Italy 0.05, Spain 0.09, Slovakia 0.21, Bulgaria 0.23, Romania 0.27, Colombia 0.33, Tanzania 0.89 and Nepal 0.84.

The result is right, however, by utilizing more specified fuzzy sets than those obtained by simple scaling would result even more divergence in the group.

Pulse	Blood Lac.inc.	Vent.inc.	Oxygen Up.inc.	Tot.sim.
156	0.8	0	0	0.27
161	1	0.04	0.17	0.40
169	1	1	1	1
174	1	0.96	1	0.97

The maximal fuzzy similarity is obtained at the value 169 beats/min, while the value 174 beats/min is quite good, too. Since there is always uncertainty involved in the measurements, sports medicine specialists would drive into a conclusion: Anaerobic threshold is close to 170 beats/min.

4.4 Classification Tasks

Theory of total fuzzy similarity offers us a powerful tool to handle with partial similar objects. For example, consider the following table of information which was collected from The World Almanac and Book of Facts 1998.

State	GDT	area	pop	bir	mor	life	65	lite	tel	car
Finland	18.2	130,1	5.1	11	5	74	14	100	1.8	2.7
Denmark	21.7	16,6	5.3	12	5	74	15	100	1.6	3.1
Belgium	19.5	11.8	10.2	12	6	74	16	99	2.2	2.4
France	18.7	210	58.0	11	6	75	16	99	1.8	2.4
Italy	18.7	116.3	57.5	10	7	75	17	97	2.3	1.9
Spain	14.3	195.4	39.2	10	6	75	16	97	2.6	2.8
Slovakia	7.2	18.9	5.4	13	11	69	11	100	4.8	5.4
Bulgaria	4.9	42.9	8.7	8	15	67	16	98	3.3	5.4
Romania	4.6	92.0	21.4	10	23	66	13	97	7.6	10.7
Colombia	5.3	440.8	37.4	21	25	70	5	91	10.0	32.5
Tanzania	0.8	364.0	29.5	41	105	40	3	68	328	589
Nepal	1.2	56.8	22.6	37	77	54	3	28	276	-

GDP = per capita GDP \$ 1000
area = area sq.mi
pop = population 10^6
bir = births per 1000 pop
mor = infant mortality/1000 live births
life = life expect. at birth
65 = age distrib. % 65+
lite = literacy%
tel = 1 telephone per x persons
car = 1 car per x persons

We may express the information of this table by fuzzy set in various ways. For example, corresponding to the first column, we may construct a fuzzy set `High GDT` by scaling, i.e.

$$\mu_{\texttt{High GDT}}(\text{state x}) = \frac{(\text{GDT of state x}) - (\text{Lowes GDT})}{(\text{Highest GDT}) - (\text{Lowest GDT})}$$

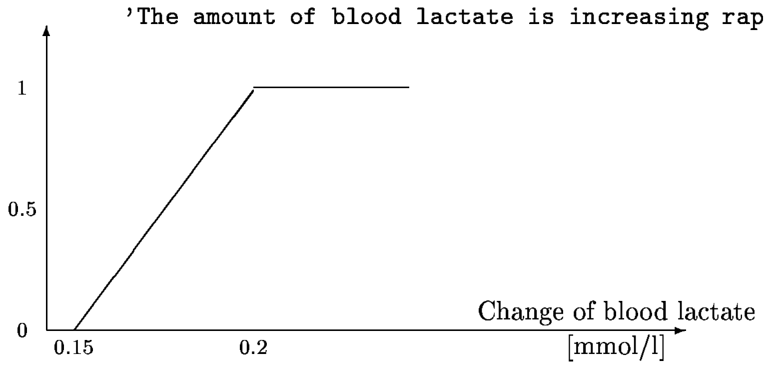
ones will have a reasonable degree of membership. In a similar manner we define another fuzzy set called

`Oxygen uptake is decreasing.`

The last fuzzy set we need is called

`The amount of blood lactate is increasing rapidly`

and is context independent. It has the following shape

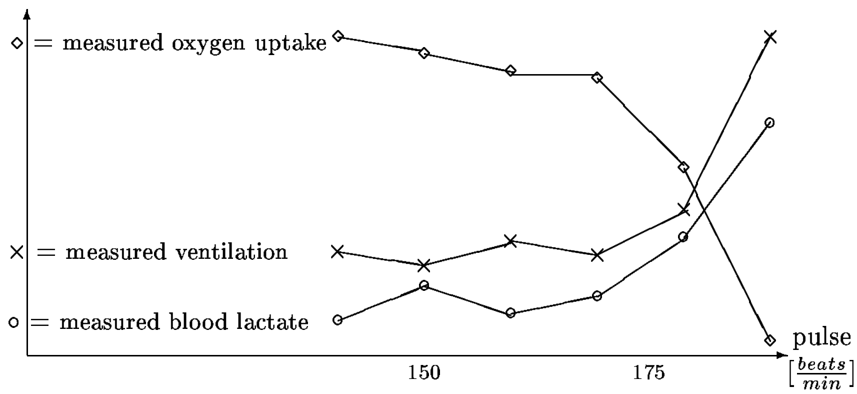


We assume there is an 'ideal object' a which belongs to all these fuzzy sets at the highest degree; we compare each measured value x and the values obtained by the spline functions with a; the one(s) which has (have) the highest degree of total similarity will be the anaerobic threshold(s) if it (they) fulfils the last crisp criteria 'pulse is 10 to 30 beats/min less than maximal pulse'. It may happen, however, that even the found highest degree of total fuzzy similarity is too low, or that there are too many possible anaerobic thresholds to draw a reliable conclusion. This fact is informed to the user with a recommendation that something went wrong and the test should be repeated.

Example Having the following test results, define the anaerobic threshold

Pulse	Δ blood lactate	Δ ventilation	Δ oxygen uptake
142	0.14	+21	+0.1
147	0.13	-3	+0.1
156	0.19	-2	-0.1
161	0.20	+1	-0.2
169	0.29	+27	-0.7
174	0.30	+26	-0.7
181	0.35	+26	-0.5
186	0.37	+27	-0.6

For simplicity, we consider only the measured values and calculate the corresponding fuzzy sets and the degrees of total fuzzy similarity. In this test the measured maximal pulse is 186 beats/min so, to fulfill the crisp rule 'pulse is 10 to 30 beats/min less than maximal pulse', we do not have to consider all the cases. We obtain the following fuzzy sets and results energy is yielded mostly aerobically, but when approaching maximal exercise level, the aerobic process with increasing lactate production start to play a more perceptible role. To guide successfully athlete's training programs, it is therefore of importance to be able to identify his aerobic and anaerobic thresholds, which are functions of blood lactate, ventilation and oxygen uptake. The test protocol of a continuous incremental exercise, which is performed e.g. by bicycle ergometer, starts with a 3 minutes warm up, then the load is increased every second minute and blood lactate, ventilation and oxygen uptake are measured. The planned duration of the test is about 20-25 minutes, the test is carried out until volitional exhaustion so usually there are $x_1, \cdots, x_n$ measurements, where $n = 10 \cdots 12$. Here is a part of a possible test result:



According to skilled sports medicine specialists, the *anaerobic threshold* is such a pulse (beats/min) that

- the amount of blood lactate is increasing rapidly (more that 0.2 mmol/l)
- ventilation is increasing clearly
- oxygen uptake is decreasing
- pulse is 15-25 (+-5) beats/min less than maximal pulse.

For example, in the case above, the anaerobic threshold would be about 170 beats/min. We construct[3] an expert system based on total fuzzy similarity to imitate sports medicine specialists' reasoning. First we connect the measured values x_i with lines as done above (in fact, they are first order spline functions!). Then, corresponding to each test, we create a fuzzy set called

`Ventilation is increasing clearly`

such that the x_i possessing the absolutely highest positive change of ventilation will have the membership degree 1, the x_j with the absolutely lowest positive change or non-positive change will have the membership degree 0 and all the other x_i:s with positive change will have a linearly scaled degree of membership in this fuzzy set. Due to the spline function, also the values in between measured

[3] A prototype can be found in [5]

	Total wait time V(C) [10 sec]															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
over saturated	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	1.0	1.0	1.0	1.0
more than medi	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	1.0	1.0	1.0	1.0	1.0
any	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Corresponding to Step 3 of the Algorithm, if the maximal total similarity is not unique, the phase with the longest waiting time will be fired, or in the worst case, the next phase will not be skipped. The performance of the fuzzy phase control is now straightforward; for example, after phase A, if there are 7 vehicles in phase B and 3 vehicles in phase C, then the next phase will be B.

HUTSIM traffic simulator simulated the performance of this control system, constructed at Helsinki University of Technology, and the results were compared to those determined by a fuzzy phase control based on a Mamdani-style fuzzy controller and a non-fuzzy control algorithm. The average waiting time per vehicle turned out to be the shortest in fuzzy similarity based control as can be seen in Table 1, Table 2 and Table 3.

veh/hour	200	400	600	800	1000	1200	1400	1600
Tot.sim	12.2	12.2	12.6	13.1	12.9	13.8	14.6	15.5
Mamdani	12.1	13.0	12.7	14.9	14.4	15.7	17.2	17.6
Non-fuz	12.1	12.9	13.5	13.9	14.6	16.1	17.2	17.5

Table 1. Average waiting time/vehicle. Vehicle flow ratio 10:1.

veh/hour	200	400	600	800	1000	1200	1400	1600
Tot.sim	12.9	12.7	13.2	12.7	12.8	13.9	15.8	17.2
Mamdani	12.7	13.0	13.7	14.0	14.2	15.3	16.8	21.5
Non-fuz	13.7	12.8	13.3	13.1	14.3	15.3	17.8	19.9

Table 2. Average waiting time/vehicle. Vehicle flow ratio 10:2.

veh/hour	200	400	600	800	1000	1200	1400	1600
Tot.sim	11.2	11.9	12.4	13.4	14.1	18.0	17.4	20.5
Mamdani	11.7	13.2	13.5	13.9	14.0	18.1	18.1	22.0
Non-fuz	12.4	11.9	13.4	14.1	14.2	17.6	18.3	77.6

Table 3. Average waiting time/vehicle. Vehicle flow ratio 10:5.

4.3 Determining Athlete's Anaerobic Thresholds

The second example on how to utilize the Algorithm takes us to the realm of sports medicine. The maximal performance capacity is essential in many sports like football, while in some other sports like long distance cycle racing the sub-maximal endurance capacity play a more important role. At low exercise levels however, if there is low request, i.e. very few or no vehicles in the next phase B or C, then this phase can be skipped. Thus, the order can be e.g. A-C-A-B-C or A-B-A-B-C. The task is to determine the right phase order; fuzzy phase selector - imitating traffic policeman's action - decides the next signal group.

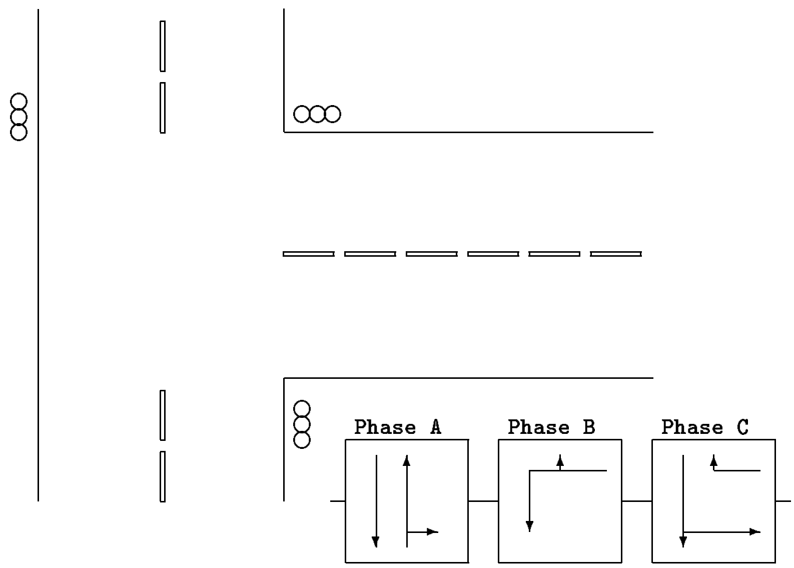


Figure 1. Layout and three phases of a T-junction

The basic principle is that phase B can be skipped if there is no request or if total waiting time of vehicles V(B) in phase B is low, and similarly, phase C can be skipped if there is no request or if total waiting time of vehicles V(C) in phase C is low. Therefore, after phase B the next phase is C or A, and after phase C the next phase is A. In details, the dynamics of the inference is the following

After phase A,

IF V(B) is high	AND V(C) is any	THEN phase is B
IF V(B) is medium	AND V(C) is over saturated	THEN phase is C
IF V(B) is low	AND V(C) is more than medium	THEN phase is C
IF V(B) is less than low	AND V(C) is more than medium	THEN phase is C

The corresponding fuzzy sets are defined by the following membership functions

	Total wait time V(B) [10 sec]															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
less than low	1.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
low	0.0	0.3	0.7	1.0	0.7	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
medium	0.0	0.0	0.0	0.0	0.5	1.0	1.0	1.0	1.0	1.0	0.5	0.0	0.0	0.0	0.0	0.0
high	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.3	0.5	0.7	0.8	1.0	1.0	1.0	1.0

Step 1. Create the dynamics of S, i.e. define the IF-THEN rules, give the shapes of the input fuzzy sets (e.g. $A_1, \cdots, C_k$) and the shapes of the output fuzzy sets (e.g. $D_1, \cdots, D_k$).

Step 2. Give weights to various parts of the input fuzzy sets (e.g. to A_i.s, B_i.s and C_i.s) to emphasize the mutual importance of the corresponding input variables.

Step 3. Put the IF-THEN-rules in a linear order with respect to their mutual importance, or give some criteria on how this can be done when necessary.

Step 4. For each THEN-part i, give a criteria on how to distinguish outputs with equal degree on membership (e.g. w_0 and v_0 such that $\mu_{D_i}(w_0) = \mu_{D_i}(v_0)$, $w_0 \neq v_0$).

A general framework for the inference system is now ready. Assume then that we have actual input values, e.g. (x_0, y_0, z_0). The corresponding output value w_0 is found in the following way.

Step 5. Consider each IF-part of the rule base as a crisp case, and compare the actual input values separately with each IF-part, in other words, count total fuzzy similarities between the actual inputs and each IF-part of the rule base; by the above Theorems, this is equivalent to counting weighted means, e.g.

$$\begin{array}{lcl} m_1\mu_{A_1}(x_0) + m_2\mu_{B_1}(y_0) + m_3\mu_{C_1}(z_0) & = & \text{Similarity(actual,Rule 1)} \\ m_1\mu_{A_2}(x_0) + m_2\mu_{B_2}(y_0) + m_3\mu_{C_2}(z_0) & = & \text{Similarity(actual,Rule 2)} \\ \vdots & & \vdots \\ m_1\mu_{A_k}(x_0) + m_2\mu_{B_k}(y_0) + m_3\mu_{C_k}(z_0) & = & \text{Similarity(actual,Rule } k\text{)} \end{array}$$

where m_1, m_2 and m_3 are the weights given in Step 2.

Step 6. Fire an output value w_0 such that

$$\mu_{D_i}(w_0) = \text{Similarity(actual,Rule } i\text{)}$$

corresponding to the maximal total fuzzy similarity Similarity(actual,Rule i), if such Rule i is not unique, use the mutual order given in Step 3, and if there are several such output values w_0 utilize the criteria given in Step 4.

Of course, we can specify our algorithm by putting extra demands, for example, in some cases the degree of total fuzzy similarity of the best alternative should be greater than some fixed value $\alpha \in [0, 1]$, sometimes all the alternatives possessing the highest fuzzy similarity should be indicated, or the difference between the best candidate and second one should be larger than a fixed value $\beta \in [0, 1]$. All this depends on an expert's choice.

4.2 Multi-phase Vehicle Control

The first example to illustrate the Algorithm origins from traffic signal control. Consider a T-junction[2] (Figure 1), where traffic flow on the main street (phase A) is assumed to be from two to ten times more intensive than traffic flow from the other direction. Normally the green traffic signal phase order is A-B-C-A,

[2] This traffic signal control system is operating in Kontula, Helsinki

$$
\begin{array}{rcl}
A & = & (\frac{1}{n}\Sigma_{i=1}^{n} S_i\langle x,y\rangle) \odot (\frac{1}{n}\Sigma_{i=1}^{n} S_i\langle y,z\rangle) \\
& = & \frac{1}{n}(\Sigma_{i=1}^{n} S_i\langle x,y\rangle + \Sigma_{i=1}^{n} S_i\langle y,z\rangle - n) \\
& = & \frac{1}{n}[(S_1\langle x,y\rangle + S_1\langle y,z\rangle - 1) + \cdots + (S_n\langle x,y\rangle + S_n\langle y,z\rangle - 1)] \\
& \leq & \frac{1}{n}(S_1\langle x,z\rangle + \cdots + (S_n\langle x,z\rangle) \\
& = & S\langle x,z\rangle
\end{array}
$$

thus, S is weakly transitive, and therefore a Lukasiewicz valued fuzzy similarity on X. The other part is now an easy generalization of this result. The proof is complete.

Theorem 2 does not hold for other BL-algebras than Lukasiewicz algebra. Indeed, consider the following two fuzzy similarities S_1 and S_2 on a set $\{a, b, c\}$ (with respect to any BL-algebra on the real unit interval!), defined by

S_1	a	b	c
a	1	1	0
b	1	1	0
c	0	0	1

and

S_2	a	b	c
a	1	0	0
b	0	1	1
c	0	1	1

The combined fuzzy relation is not a fuzzy similarity on the set $\{a, b, c\}$ if one uses Gödel algebra or Product t-algebra; in general, weak transitivity does hold.

4.1 Algorithm to Construct Fuzzy Inference Systems

A control situation comprehends a system S, an input universe of discourse X, the IF-parts, and an output universe of discourse Y, the THEN-parts. We assume there are n input variables and one output variable. The dynamics of S are characterized by a finite collection of IF-THEN-rules; e.g.

Rule 1 IF x is A_1 and y is B_1 and z is C_1 THEN w is D_1
Rule 2 IF x is A_2 and y is B_2 and z is C_2 THEN w is D_2
⋮ ⋮
Rule k IF x is A_k and y is B_k and z is C_k THEN w is D_k

where $A_1, \cdots, D_k$ are fuzzy sets of height 1, that is, in each fuzzy set there is at least one element that obtains the membership degree 1. Generally, the output fuzzy sets $D_1, \cdots, D_k$ should obtain all the same values $\in [0,1]$ the input fuzzy sets $A_1, \cdots, C_k$ do, however, the outputs can be crisp actions, too. All these fuzzy sets are to be specified by the fuzzy control engineer. We avoid disjunction between the rules by allowing some of the output fuzzy sets D_i and $D_j, i \neq j$, be possibly equal . Thus, a fixed THEN-part can follow various IF-parts. Some of the input fuzzy sets may be equal, too (e.g. $B_i = B_j$ for some $i \neq j$). However, the rule base should be consistent; a fixed IF-part precedes a unique THEN-part. Moreover, the rule base can be incomplete; if an expert is not able to define the THEN-part of some combination 'IF x is A_i and y is B_i and z is C_i' then the rule should be skipped.

Now we are in the position to formulate an algorithm a fuzzy control engineer has to perform to construct a total fuzzy similarity based inference system.

Definition 10 *Let A be a non-void set and $\odot$ a continuos t-norm. Then a* fuzzy similarity *S on A is such a binary fuzzy relation that, for each $x, y, z \in A$,*
(i) $S\langle x, x\rangle = 1$ (everything is similar to itself),
(ii) $S\langle x, y\rangle = S\langle y, x\rangle$ (fuzzy similarity is symmetric),
(iii) $S\langle x, y\rangle \odot S\langle y, z\rangle \leq S\langle x, z\rangle$ (fuzzy similarity is weakly transitive).

Trivially, fuzzy similarity is a generalization of classical equivalence relation, thus called *many-valued equivalence*, too. Notice that, by weak transitivity, partial similarity of x and y, and y and z imply only a lower bound for the degree of similarity of x and z.

Recall a *fuzzy set* X is an ordered couple (A, μ_X), where the *reference set* A is a non-void set and the *membership function* $\mu_X : A \searrow [0, 1]$ tells the degree to which an element $a \in A$ belongs to the fuzzy set X.

Theorem 2 *Any fuzzy set (A, μ_X) on a reference set A generates a fuzzy similarity S on A, defined by*

$$S(x, y) = \mu_X(x) \leftrightarrow \mu_X(y), \text{ where } x, y \text{ are elements of } A.$$

Moreover,

$$\text{if } \mu_X(y) = 1 \text{ then } S(x, y) = \mu_X(x).$$

Proof. By (38)-(41).

It is worth noting that, in Lukasiewicz-Pavelka logic, 'the negation of equivalence is distance'. Indeed, for all $a, b \in [0, 1]$,

$$(a \leftrightarrow b)^* = 1 - [1 - |a - b|] = |a - b|,$$

the Euclidean distance between a and b.

Theorem 3 *Consider n Łukasiewicz valued fuzzy similarities S_i, $i = 1, \cdots, n$ on a set X. Then*

$$S\langle x, y\rangle = \tfrac{1}{n}\Sigma_{i=1}^{n} S_i\langle x, y\rangle$$

is a Łukasiewicz valued fuzzy similarity on X. More generally, the weighted mean

$$S\langle x, y\rangle = \tfrac{1}{M}\Sigma_{i=1}^{n} m_i \cdot S_i\langle x, y\rangle,$$

where $M = \Sigma_{i=1}^{n} m_i, m_i \in \mathcal{N}$, is again a Łukasiewicz valued fuzzy similarity on X, called total fuzzy similarity relation.

Proof. Since all S_i, $i = 1, \cdots, n$ are reflexive and symmetric so is S. The weak transitivity of S can be seen in the following way. Let $A = S\langle x, y\rangle \odot S\langle y, z\rangle$. If $A = 0$, then trivially $A \leq S\langle x, z\rangle$, therefore assume $A > 0$. Then

4 Similarity-based reasoning

The objective in approximate reasoning is to draw conclusions from partially true premises. Our idea is to look for the most similar premise, the IF-part, and fire the corresponding conciliation, the THEN-part. Moreover, the degree of similarity may be composed of various partial similarities. Lukasiewicz-Pavelka logic provides a reasonable method to do this task, in a sense Łukasiewicz-Pavelka logic is the only many-valued logic to come off this challenge as we shall now see.

Recall a binary operation $\odot : [0,1]^2 \searrow [0,1]$ is called *t-norm* if, for all elements $x, y, z \in [0,1]$, (i) if $x \leq y$, then $x \odot z \leq y \odot z$, (ii) $x \odot y = y \odot x$, (iii) $x \odot 1 = x$, (iv) $x \odot (y \odot z) = (x \odot y) \odot z$, (v) $x \odot 0 = 0$.

In particular, *continuous* t-norms $\odot$ and their residua $\rightarrow$ play a fundamental role in fuzzy logic. The most frequently used continuos t-norms in various fuzzy inference systems generate the following algebraic structures

Gödel algebra:

$$x \odot y = \min\{x, y\},\ x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$$

Product t-algebra:

$$x \odot y = xy,\ x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

Łukasiewicz algebra:

$$x \odot y = \max\{0, x + y - 1\},\ x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ 1 - x + y & \text{otherwise.} \end{cases}$$

These three examples are fundamental since, in a certain sense, they characterize all possible continuos t-norms (for details, see [3],[4]). They are the generators of all *BL-algebras* of the real unit interval, too; by fixing a continuous t-norm we fix a *Basic Logic*, a well-defined many-valued logic modeling mathematically fuzzy reasoning. Łukasiewicz-Pavelka logic is, in particular, a Basic Logic. The operations $\odot$ and $\rightarrow$ are the algebraic counterparts of the logical connectives conjunction and implication, respectively. In particular, the complement x^* of an element $x \in [0,1]$ defined by $x^* = x \rightarrow 0$, is the algebraic counterpart of negation, while many-valued equivalence is interpreted algebraically by *bi-residuum* defined, for all $x, y \in [0,1]$, via

$$x \leftrightarrow y = \min\{(x \rightarrow y), (y \rightarrow x)\}.$$

In any BL-algebra, a bi-residuum $\leftrightarrow$ has the following properties (cf. [11])

$$x \leftrightarrow x = 1, \tag{38}$$

$$x \leftrightarrow y = y \leftrightarrow x, \tag{39}$$

$$(x \leftrightarrow y) \odot (y \leftrightarrow z) \leq x \leftrightarrow z, \tag{40}$$

$$x \leftrightarrow 1 = x. \tag{41}$$

Now we can set the following [13] important

We conclude that $T \vdash_1 \alpha$ holds for each $\alpha \in \mathcal{F}$. Conversely, if $T \vdash_1 \alpha$ holds for each $\alpha \in \mathcal{F}$, then, in particular, $T \vdash_1 \mathbf{0}$, i.e. $C^{\mathtt{syn}}(T)(\mathbf{0}) = \mathbf{1} \neq \mathbf{0}$.

Proposition 4 *A fuzzy theory T is contradictory iff the following condition holds* (C)*:*

There is a formula α and metaproofs w, w' for α, $\mathtt{non}\text{-}\alpha$, respectively, such that $Val_T(w) = a$, $Val_T(w') = b$ and $\mathbf{0} < a \odot b$.

Let T be a fuzzy theory. The choice of the logical axioms Ax.1 - Ax.11 and soundness of fuzzy rules of inference guarantee, for each formula α, each metaproof w for α in T, each valuation v which satisfies T, that $Val_T(w) \leq v(\alpha)$. Thus,

$$\bigvee\{Val_T(w) \mid w \textit{ is a metaproof for } \alpha \textit{ in } T\} \leq \bigwedge\{v(\alpha) \mid v \textit{ satisfies } T\},$$

by symbols, $\mathcal{C}^{\mathtt{syn}}(T)(\alpha) \leq \mathcal{C}^{\mathtt{sem}}(T)(\alpha)$. (This (in-)equality holds even if T is not satisfiable as $\bigwedge\{\emptyset\} = \mathbf{1}$.) We write

Theorem 1 (Soundness Theorem for Fuzzy Propositional Calculus) *Let T be a fuzzy theory. For each formula α, if $T \vdash_a \alpha$, $T \models_b \alpha$, then $a \leq b$.*

Corollary 1 *Any satisfiable fuzzy theory T is consistent.*

The most important theoretical result concerning Lukasiewicz-Pavelka logic is the *Completeness Theorem* of Fuzzy Propositional Calculus; for any fuzzy theory T, for each formula $\alpha \in \mathcal{F}$ and for any value $a \in L$, holds

$$T \vdash_a \alpha \text{ if, and only if } T \models_a \alpha.$$

The rather long proof of this fact is, however, omitted. We conclude this section by giving an easy example, which should illustrate a possible application of Lukasiewicz-Pavelka logic.

Example Assume p stands for *It is raining enough* and q stands for *Potato is growing fast.* We study a fuzzy theory T such that
$T(\mathtt{non} - p \ \mathtt{imp} \ \mathtt{non} - q) = 1$ standing for *If it is not raining enough then potato is not growing fast* and $T(q) = 0.7$ standing loosely for *Potato is growing more or less fast.* Now we are interested in the degree of deduction of p. We find the following metaproof for p:

$(\mathtt{non} - p \ \mathtt{imp} \ \mathtt{non} - q) \ \mathtt{imp} \ (q \ \mathtt{imp} \ p)$	,	1	,	$Ax.11$
$(\mathtt{non} - p \ \mathtt{imp} \ \mathtt{non} - q)$	,	1	,	*non-logical axiom*
$(q \ \mathtt{imp} \ p)$	,	1	,	R_{GMP}
q	,	0.7	,	*non-logical axiom*
p	,	0.7	,	R_{GMP}

Therefore $0.7 \leq \mathcal{C}^{\mathtt{syn}}(p)$. Since a valuation v such that $v(p) = v(q) = 0.7$ satisfies T, we have, by Completeness Theorem, $0.7 \leq \mathcal{C}^{\mathtt{syn}}(T)(p) = \mathcal{C}^{\mathtt{sem}}(T)(p) \leq 0.7$. Thus, the degree of deduction of p is 0.7. Freely speaking, *Is is raining more or less enough.*

A *metaproof* of a well-formed formula α in a fuzzy theory $\langle \mathtt{A}, \mathtt{R}, T\rangle$, denoted by w, is a finite sequence

$$\begin{array}{ccc} \alpha_1 & , & a_1 \\ \vdots & & \vdots \\ \alpha_m & , & a_m \end{array}$$

of pairs $(\alpha_i, a_i) \in \mathcal{F} \times L$ such that the following holds: (i) $\alpha_m = \alpha$, (ii) for each $i, 1 \leq i \leq m, \alpha_i$ is a logical axiom, or α_i is a non-logical axiom, or there are a fuzzy rule of inference in $\mathtt{R}$ and formulas $\alpha_{i_1}, \cdots, \alpha_{i_n}$ with $i_1, \cdots, i_n < i$ such that $\alpha_i = r^{\mathtt{syn}}(\alpha_{i_1}, \cdots, \alpha_{i_n})$, (iii) for each $i, 1 \leq i \leq m$, the value a_i is given by

$$a_i = \begin{cases} a & \text{if } \alpha_i \text{ is the axiom } \mathbf{a} \\ \mathbf{1} & \text{if } \alpha_i \text{ is some other logical axiom} \\ T(\alpha_i) & \text{if } \alpha_i \text{ is a non-logical axiom} \\ r^{\mathtt{sem}}(a_{i_1}, \cdots, a_{i_n}) & \text{if } \alpha_i = r^{\mathtt{syn}}(\alpha_{i_1}, \cdots, \alpha_{i_n}). \end{cases}$$

The value a_m is denoted by $Val_{\langle \mathtt{A},\mathtt{R}, T\rangle}(w)$ and is called the *degree* of the metaproof w. Because a formula α may have many metaproofs with different degrees, we define the *degree of deduction* of the formula α in fuzzy theory $\langle \mathtt{A}, \mathtt{R}, T\rangle$ by

$$C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(T)(\alpha) = \bigvee\{ Val_{\langle \mathtt{A},\mathtt{R}, T\rangle}(w) | w \text{ is a metaproof for } \alpha \text{ in } \langle \mathtt{A}, \mathtt{R}, T\rangle\}.$$

The case $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}(T)(\alpha) = a$ is denoted by $\langle \mathtt{A}, \mathtt{R}, T\rangle \vdash_a \alpha$, in particular, $\vdash_a \alpha$ if the set of logical axioms $\mathtt{A}$ is composed of Ax.1 - Ax.11, the set of fuzzy rules of inference $\mathtt{R}$ contains R_{GMP}, R_{a-CTR}, R_{a-LR}, R_{RBC} and T is the void set.

Let the set of logical axioms $\mathtt{A}$ be composed of Ax.1 - Ax.11, and the set of fuzzy rules of inference $\mathtt{R}$ contains the fuzzy rules of inference R_{GMP}, R_{a-CTR}, R_{a-LR}, R_{RBC}. Fuzzy theories are thus identified by means of their sets T of non-logical axioms. We will write $C^{\mathtt{syn}}$ instead of $C^{\mathtt{syn}\langle \mathtt{A},\mathtt{R}\rangle}$. Obviously, for any fuzzy theory T, if $\vdash_1 \alpha$ then $T \vdash_1 \alpha$, and by Ax.7, for any inner truth value $\mathbf{a}$, $a \leq C^{\mathtt{syn}}(T)(\mathbf{a})$. This leads us to the following

Definition 9 *A fuzzy theory T is* consistent *if, for any inner truth value* $\mathbf{a}$, $a = C^{\mathtt{syn}}(T)(\mathbf{a})$*, and otherwise T is* contradictory.

Proposition 3 *A fuzzy theory T is contradictory iff $T \vdash_1 \alpha$ holds for each $\alpha \in \mathcal{F}$.*

Proof. Assume T is contradictory. Then there exists an inner truth value $\mathbf{a}$ such that $a \neq C^{\mathtt{syn}}(T)(\mathbf{a})$. If for each metaproof w for $\mathbf{a}$ holds $Val_T(w) \leq a$, then $a \leq C^{\mathtt{syn}}(T)(\mathbf{a}) \leq a$, hence $C^{\mathtt{syn}}(T)(\mathbf{a}) = a$, which is not the case. Therefore there exists a metaproof w for $\mathbf{a}$ such that $Val_T(w) \not\leq a$. For every formula $\alpha \in \mathcal{F}$, we have now the following metaproof:

$$\begin{array}{lclcr} \mathbf{a} & , & Val_T(w) & , & \textit{assumption} \\ \mathbf{0} & , & \mathbf{1} & , & R_{a-CTR} \\ \mathbf{0} \texttt{ imp } \alpha & , & \mathbf{1} & , & Ax.4 \\ \alpha & , & \mathbf{1} & , & R_{GMP} \end{array}$$

Generalized Modus Ponens:

$$R_{GMP} \quad : \quad \frac{\alpha, (\alpha \texttt{ imp } \beta)}{\beta}, \frac{a, b}{a \odot b}$$

a-Consistency-testing rules:

$$R_{a-CTR} \quad : \quad \frac{\mathbf{a}}{\mathbf{0}}, \frac{b}{c}$$

where **a** *is an inner truth value, and* $c = \mathbf{0}$ *if* $b \leq a$ *and* $c = \mathbf{1}$ *elsewhere.*
a-Lifting Rules:

$$R_{a-LR} \quad : \quad \frac{\alpha}{(\mathbf{a} \texttt{ imp } \alpha)}, \frac{b}{a \to b}$$

where **a** *is an inner truth value.*
Rule of Bold Conjunction:

$$R_{RBC} \quad : \quad \frac{\alpha, \beta}{(\alpha \texttt{ and } \beta)}, \frac{a, b}{a \odot b}$$

Definition 7 *A fuzzy set* $\mathtt{A}$ *of* logical axioms *is a finite set of forms of formulas each being an inner truth value* **a**, *then* $\mathtt{A}(\mathbf{a}) = a$, *or a tautology* α *at the degree* $\mathbf{1}$, *then* $\mathtt{A}(\alpha) = \mathbf{1}$. *Elsewhere* $\mathtt{A}(\alpha) = \mathbf{0}$.

For example, the following set of forms of formulas is, by (27)-(37), a set of logical axioms:
(Ax.1) $\alpha \texttt{ imp } \alpha$,
(Ax.2) $(\alpha \texttt{ imp } \beta) \texttt{ imp } [(\beta \texttt{ imp } \gamma) \texttt{ imp } (\alpha \texttt{ imp } \gamma)]$,
(Ax.3) $(\alpha_1 \texttt{imp} \beta_1) \texttt{ imp } \{(\beta_2 \texttt{imp} \alpha_2) \texttt{ imp } [(\beta_1 \texttt{imp} \beta_2) \texttt{ imp } (\alpha_1 \texttt{imp} \alpha_2)]\}$,
(Ax.4) $\alpha \texttt{ imp } \mathbf{1}$,
(Ax.5) $\mathbf{0} \texttt{ imp } \alpha$,
(Ax.6) $[(\alpha \texttt{ and } \texttt{non-}\alpha) \texttt{ imp } \mathbf{0}$,
(Ax.7) **a**,
(Ax.8) $\alpha \texttt{ imp } (\beta \texttt{ imp } \alpha)$,
(Ax.9) $(\mathbf{1} \texttt{ imp } \alpha) \texttt{ imp } \alpha$,
(Ax.10) $[(\alpha \texttt{ imp } \beta) \texttt{ imp } \beta] \texttt{ imp } [(\beta \texttt{ imp } \alpha) \texttt{ imp } \alpha]$,
(Ax.11) $(\texttt{non-}\alpha \texttt{ imp } \texttt{non-}\beta) \texttt{ imp } (\beta \texttt{ imp } \alpha)$.
where α, β, γ, α_1, α_2, β_1, β_2 are well-formed formulas and **a** any inner truth-value. Values $\mathtt{A}(\delta)$ are obvious.

Definition 8 *Let* $\mathtt{A}$ *be a fixed set of logical axioms,* $\mathtt{R}$ *a fixed finite set of fuzzy rules of inference and* T *fuzzy set of formulas called* non-logical axioms. *Then a (zero-order)* fuzzy theory *is a triplet* $\langle \mathtt{A}, \mathtt{R}, T \rangle$. *In particular, if the set of logical axioms* $\mathtt{A}$ *is composed of Ax.1 - Ax.11, and the set of fuzzy rules of inference* $\mathtt{R}$ *contains* R_{GMP}, R_{a-CTR}, R_{a-LR}, R_{RBC}, *we denote a fuzzy theory simply by* T, *and if* T *is the void set we talk about* Fuzzy Propositional Calculus.

$\alpha_1, \cdots, \alpha_n$ $(1 \leq n)$ in a formalized language associates another formula β in this language in such a way that β is a logical consequence of the formulas $\alpha_1, \cdots, \alpha_n$. This fact is usually denoted as follows

$$\frac{\alpha_1, \cdots, \alpha_n}{\beta}$$

Formulas $\alpha_1, \cdots, \alpha_1$ are called *premises* and β the *conclusion* of this rule of inference. For example,

$$\frac{\mathtt{non} - (\mathtt{non} - \alpha)}{\alpha}$$

and

$$\frac{\alpha, (\alpha \ \mathtt{imp} \ \beta)}{\beta}$$

are rules of inference in Classical Propositional Logic, called *Rule of Double Negation* and *Modus Ponens,* respectively. By saying that a formula β is a *logical consequence* of a set S of formulas we mean that if every formula α belonging to S is acknowledged to be true, then β must be accepted as true. Thus, the most important property of rule of inference is soundness, i.e. rule of inference preserves truth.

We define a fuzzy rule of inference as consisting of two components. The first component operates on formulas and is, in fact, a rule of inference in the usual sense; the second component operates on truth values and says how the truth value of the conclusion is to be computed from the truth-values of the premises such that the degree of truth is preserved. More accurately, we set

Definition 6 *An n-ary* fuzzy rule of inference *is a scheme*

$$R \quad : \quad \frac{\alpha_1, \cdots, \alpha_n}{r^{\mathtt{syn}}(\alpha_1, \cdots, \alpha_n)}, \frac{a_1, \cdots, a_n}{r^{\mathtt{sem}}(a_1, \cdots, a_n)}$$

where the well-formed formulae $\alpha_1, \cdots, \alpha_n$ are the premises *and the well-formed formula $r^{\mathtt{syn}}(\alpha_1, \cdots, \alpha_n)$ is the* conclusion.

The values $a_1, \cdots, a_n$, $r^{\mathtt{sem}}(a_1, \cdots, a_n) \in L$ are the corresponding truth-values. The mapping $r^{\mathtt{sem}} : L^n \searrow L$ is semi-continuous *on each variable, i.e. it holds always that*

$$r^{\mathtt{sem}}(a_1, \cdots, \bigvee_{j \in \Gamma} a_{k_j}, \cdots, a_n) = \bigvee_{j \in \Gamma} r^{\mathtt{sem}}(a_1, \cdots, a_{k_j}, \cdots, a_n), 1 \leq k \leq n.$$

We assume the fuzzy rule of inference is sound, *i.e. for each valuation v holds*

$$r^{\mathtt{sem}}(v(\alpha_1), \cdots, v(\alpha_n)) \leq v(r^{\mathtt{syn}}(\alpha_1, \cdots, \alpha_n)).$$

Proposition 2 *The following schemes are fuzzy rules of inference in Łukasiewicz-Pavelka logic*

Definition 5 *The* degree of validity *of a formula $\alpha \in \mathcal{F}$ (with respect to a fuzzy set of formulas T) is a value*

$$\mathcal{C}^{\texttt{sem}}(T)(\alpha) = \bigwedge\{v(\alpha) \mid v \textit{ satisfies } T\}. \tag{25}$$

In particular, if T is the void set we define the degree of tautology *of a formula α by*

$$\mathcal{C}^{\texttt{sem}}(\alpha) = \bigwedge\{v(\alpha) \mid v \textit{ is a valuation } \}. \tag{26}$$

This definition is very natural and generalizes the concept of tautology in classical logic. If $\mathcal{C}^{\texttt{sem}}(T)(\alpha) = a$ we write $T \models_a \alpha$, in particular $\models_a \alpha$ if T is the void set. Of special interest will be formulas α such that $\models_1 \alpha$. Evidently, if $\models_1 \alpha$, then $T \models_1 \alpha$ for any fuzzy set T of formulas (even for those T which are not satisfied by any valuation v!)

Proposition 1 *Let α, β, γ, α_1, β_1, α_2, β_2 be formulas and* **c** *any inner truth value. Then the following forms of formulas are universally valid at the degree 1, except for, of course, the inner truth value* **c***, which is universally valid at the degree c, i.e.*

$$\models_1 \alpha \texttt{ imp } \alpha, \tag{27}$$

$$\models_1 (\alpha \texttt{ imp } \beta) \texttt{ imp } [(\beta \texttt{ imp } \gamma) \texttt{ imp } (\alpha \texttt{ imp } \gamma)], \tag{28}$$

$$\models_1 (\alpha_1 \texttt{imp} \beta_1) \texttt{ imp } \{(\beta_2 \texttt{imp} \alpha_2) \texttt{ imp } [(\beta_1 \texttt{imp} \beta_2) \texttt{ imp } (\alpha_1 \texttt{imp} \alpha_2)]\}, \tag{29}$$

$$\models_1 \alpha \texttt{ imp } \mathbf{1}, \tag{30}$$

$$\models_1 \mathbf{0} \texttt{ imp } \alpha, \tag{31}$$

$$\models_1 (\alpha \texttt{ and } \texttt{ non-}\alpha) \texttt{ imp } \mathbf{0}, \tag{32}$$

$$\models_c \mathbf{c}, \tag{33}$$

$$\models_1 \alpha \texttt{ imp } (\beta \texttt{ imp } \alpha), \tag{34}$$

$$\models_1 (\mathbf{1} \texttt{ imp } \alpha) \texttt{ imp } \alpha, \tag{35}$$

$$\models_1 [(\alpha \texttt{ imp } \beta) \texttt{ imp } \beta] \texttt{ imp } [(\beta \texttt{ imp } \alpha) \texttt{ imp } \alpha], \tag{36}$$

$$\models_1 (\texttt{non-}\alpha \texttt{ imp } \texttt{ non-}\beta) \texttt{ imp } (\beta \texttt{ imp } \alpha). \tag{37}$$

The definitions of valuation and the degree of validity of formula α are natural and relatively easy generalizations of the corresponding concepts in two valued logic. Now we consider the following related non-trivial problem

Knowing that a formula α is valid at a certain degree, do there exist a fuzzy set of axioms and fuzzy rules of inference by which we can infer α at the same degree?

In other words, is Łukasiewicz-Pavelka logic axiomatizable? To find an answer to this question, we start by defining on what we mean by fuzzy axiom, fuzzy rule of inference, fuzzy proof, etc.

A *rule of inference* in Classical Propositional Logic is an n-ary operation on the set of well-formed formulas which with a finite sequence of formulas even if the abbreviation of the logical connective $\overline{\texttt{or}}$ is far from being obvious. We also introduce a logical connective `equiv` by abbreviating

$$(\alpha \texttt{ equiv } \beta) = [(\alpha \texttt{ imp } \beta) \overline{\texttt{and}} [(\beta \texttt{ imp } \alpha)],$$

thus generalizing the situation in classical logic. Then we have, for any valuation v, any formulas $\alpha, \beta \in \mathcal{F}$,

$$v(\alpha \texttt{ equiv } \beta) = v(\alpha) \leftrightarrow v(\beta).$$

In classical logic truth value functions $v : \mathcal{F} \searrow \{0,1\}$ satisfy the truth tables

$v(\alpha \texttt{ or } \beta)$	$v(\beta) = 1$	$v(\beta) = 0$
$v(\alpha) = 1$	1	1
$v(\alpha) = 0$	1	0

$v(\alpha \texttt{ and } \beta)$	$v(\beta) = 1$	$v(\beta) = 0$
$v(\alpha) = 1$	1	0
$v(\alpha) = 0$	0	0

$v(\alpha \texttt{ imp } \beta)$	$v(\beta) = 1$	$v(\beta) = 0$
$v(\alpha) = 1$	1	0
$v(\alpha) = 0$	1	1

	$v(\alpha) = 1$	$v(\alpha) = 0$
$v(\texttt{non-}\alpha)$	0	1

It is easy to verify that fuzzy valuations satisfy these tables and that in two valued case the truth value tables of the logical connectives $\overline{\texttt{and}}$ and `and` as well as $\overline{\texttt{or}}$ and `or` coincide. Complete truth tables are not, of course, possible in logic with infinite many values of truth. We may, however, write instances of them.

As an example, we calculate that if $v(\alpha) = 0.2$, $v(\beta) = 0.6$ and $v(\gamma) = 0.9$ then

$$v([\alpha \texttt{ imp } (\texttt{non-}\beta \texttt{or non-}\gamma)] \texttt{ imp } \gamma) = 0.9$$

In logic we are interested in the logical consequences of given statements. From semantic point of view this raises a question

Associating fixed values of truth to a set of well-formed formulas $T \subseteq \mathcal{F}$, what is the least degree of truth, or greatest lower bound of such degrees, of an arbitrary formula $\alpha \in \mathcal{F}$ with respect to T?

This leads us to the following semantic definitions

Definition 4 *A* fuzzy set T of formulas *is a function* $T : \mathcal{F} \searrow L$. *A truth value function* $v : \mathcal{F} \searrow L$ satisfies T *if* $T(\alpha) \leq v(\alpha)$ *for any formula* $\alpha \in \mathcal{F}$. *If there exists a valuation* v *such that* v *satisfies* T *the* T *is called* satisfiable.

The void set $\emptyset$ can be regarded as a fuzzy set of formulas by defining $\emptyset(\alpha) = 0$ for all formulas $\alpha \in \mathcal{F}$. The void set is of course satisfiable.

Inner truth-values and propositional variables are *atomic formulas.*

Definition 2 *The set $\mathcal{F}$ of well-formed formulas is constructed in the following way:*
(i) *atomic formulas are in $\mathcal{F}$,*
(ii) *if α and β are in $\mathcal{F}$ then $(\alpha \texttt{ imp } \beta)$ and $(\alpha \texttt{ and } \beta)$ are in $\mathcal{F}$.*

Propositional variables correspond statements like *it is raining*, etc. The main difference between two-valued sentential logic and Lukasiewicz-Pavelka logic are the inner truth-values, which can be regarded as generalizations of the falsum sign $\perp$ of classical logic. We follow the idea of intuitionistic logic and abbreviate $(\alpha \texttt{ imp } \mathbf{0})$ by $(\texttt{non} - \alpha)$. The logical connective non (read 'non') is called *negation.* Giving *semantic interpretation* to a formula $\alpha \in \mathcal{F}$ means we associate a value of truth $v(\alpha) \in L$ to α, in other words, we define *truth value function* $v : \mathcal{F} \searrow L$ by setting

Definition 3 *A function $v : \mathcal{F} \searrow L$ such that, for any inner truth value* **a** *and for any formulas α, β,*

$$v(\mathbf{a}) = a, \tag{20}$$
$$v(\alpha \texttt{ imp } \beta) = v(\alpha) \to v(\beta), \tag{21}$$
$$v(\alpha \texttt{ and } \beta) = v(\alpha) \odot v(\beta), \tag{22}$$

is called (fuzzy) valuation *or* (fuzzy) truth value function.

We introduce a logical connective or (read 'or' or 'disjunction') as an abbreviation

$$(\alpha \texttt{ or } \beta) = [\texttt{non-(non-}\alpha \texttt{ and non-}\beta)].$$

This generalizes the state of affairs in classical sentential logic and makes the formalized language of many-valued logic easier to read.

Remark 1 *For any valuation v, any $\alpha, \beta \in \mathcal{F}$,*

$$v(\texttt{non-}\alpha) = v(\alpha)^*, \tag{23}$$
$$v(\alpha \texttt{ or } \beta) = v(\alpha) \oplus v(\beta). \tag{24}$$

Generally, $v(\alpha \texttt{ and } \beta) < v(\alpha) \wedge v(\beta)$ and $v(\alpha) \vee v(\beta) < v(\alpha \texttt{ or } \beta)$. In application we may need, however, disjunctive and conjunctive connectives, denote them by $\overline{\texttt{and}}$ and $\overline{\texttt{or}}$, respectively, such that

$$v(\alpha \overline{\texttt{and}} \beta) = v(\alpha) \wedge v(\beta),\ v(\alpha \overline{\texttt{or}} \beta) = v(\alpha) \vee v(\beta).$$

Abbreviating can do this

$$(\alpha \overline{\texttt{or}} \beta) = [(\alpha \texttt{ imp } \beta) \texttt{ imp } \beta)]$$
$$(\alpha \overline{\texttt{and}} \beta) = [\texttt{non-(non-}\alpha \overline{\texttt{or}} \texttt{ non-}\beta)],$$

the following equations - the original MV-algebra axioms by Chang [1]- are satisfied in every Wajsberg algebra.

$$x \oplus y = y \oplus x \quad , \quad x \odot y = y \odot x, \tag{10}$$
$$x \oplus (y \oplus z) = (x \oplus y) \oplus z \quad , \quad x \odot (y \odot z) = (x \odot y) \odot z, \tag{11}$$
$$x \oplus x^* = \mathbf{1} \quad , \quad x \odot x^* = \mathbf{0}, \tag{12}$$
$$x \oplus \mathbf{1} = \mathbf{1} \quad , \quad x \odot \mathbf{0} = \mathbf{0}, \tag{13}$$
$$x \oplus \mathbf{0} = x \quad , \quad x \odot \mathbf{1} = x, \tag{14}$$
$$(x \oplus y)^* = x^* \odot y^* \quad , \quad (x \odot y)^* = x^* \oplus y^*, \tag{15}$$
$$x^{**} = x \quad , \quad \mathbf{1}^* = \mathbf{0}, \tag{16}$$
$$x \vee y = y \vee x \quad , \quad x \wedge y = y \wedge x, \tag{17}$$
$$x \vee (y \vee z) = (x \vee y) \vee z \quad , \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z, \tag{18}$$
$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z) \quad , \quad x \odot (y \vee z) = (x \odot y) \vee (x \odot z). \tag{19}$$

Conversely, given an MV-algebra $\langle L, \oplus, \odot, ^*, \mathbf{0}, \mathbf{1}\rangle$, we can define an operation $\rightarrow$ for all $x, y \in L$ by $x \rightarrow y = x^* \oplus y$. Then the Wajsberg algebra axioms hold. Thus, there is a one-to-one correspondence between MV-algebras and Wajsberg algebras. An MV-algebra is called *complete* if it contains lowest upper bound and least lower bound of any of its subset $\{x_i \mid i \in \Gamma\}$, denoted by $\bigvee\{x_i \mid i \in \Gamma\}$ and $\bigwedge\{x_i \mid i \in \Gamma\}$, respectively.

In the real unit interval, the most used MV-algebra structure called Łukasiewicz algebra, too, is obtained by setting $1 = \mathbf{1}$, $0 = \mathbf{0}$ and, for all $x, y \in [0, 1]$,

$x \rightarrow y = \min\{1 - x + y, 1\}$, $x^* = 1 - x$
$x \odot y = \max\{x + y - 1, 0\}$, $x \oplus y = \min\{x + y, 1\}$
$x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$.

These algebraic operations will offer us an elegant tool to interpret the logical connectives implication, negation and two kinds of conjunction and disjunction, respectively. Moreover, for equivalence in Łukasiewicz-Pavelka logic we have the operation $x \leftrightarrow y = 1 - |x - y|$ which, in general setting, is defined by
$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$.

3 Łukasiewicz-Pavelka Logic

Now we start to develop logic, which allows more truth values than only 'false' and 'true'. We assume all the time that the set L of values of truth forms a complete MV-algebra. The formalized language of this logic is composed of four kinds of building blocks:
(i) The set of *propositions* is an infinite set $L = \{p_i; i \in \mathcal{N}\}$. Propositions are sometimes denoted by p, q, r, s, t, w, too.
(ii) For any element $a \in L$ there is an *inner truth value* **a** in the language.
(iii) The logical connectives are `imp` (read 'implies') and **`and`** (read 'and' or 'conjunction').
(iv) There are auxiliary symbols $\},],), (, [, \{$ in the language of the logic.

the genus but do not perceive it in others'. According to Niiniluoto [9], 'The real challenge $\cdots$ is that we have to extend our treatment from simple analogy to multiple analogy'.

We shall see how generalizing equivalence relation can solve this challenge on the basis of Lukasiewicz-Pavelka many-valued logic, and applying the generated many-valued equivalence on Zadeh's fuzzy sets. We introduce an algorithm to construct fuzzy IF-THEN inference systems having a special feature that whenever the output would not be unique the final decision should be left to human experts. Thus, as much as possible the intelligence relies on a real controller, and technical defuzzification methods are not needed. We demonstrate how approximate reasoning, many classification tasks, case-based reasoning, etc. can be viewed as applications of many-valued similarity and, thus, Lukasiewicz-Pavelka logic.

2 The Algebra of Łukasiewicz-Pavelka logic

An axiom system of a logic generates an algebraic structure. In classical logic this structure is Boolean algebra, while infinite valued Łukasiewicz-Pavelka logic, being an extension of two-valued logic, generates a more general algebraic structure, called *MV-algebra* [1],[2]. To minimize the axioms we first set, however, the following

Definition 1 *Let L be a non-void set, $\mathbf{1} \in L$ and $\rightarrow$, * be a binary and a unary operation, respectively, defined on L such that, for each $x, y, z \in L$,*

$$\mathbf{1} \rightarrow x = x, \tag{1}$$

$$(x \rightarrow y) \rightarrow [(y \rightarrow z) \rightarrow (x \rightarrow z)] = \mathbf{1}, \tag{2}$$

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x, \tag{3}$$

$$(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = \mathbf{1}. \tag{4}$$

Then the system $\langle L, \rightarrow, ^*, \mathbf{1} \rangle$ is called a Wajsberg algebra.

Now define on a Wajsberg algebra $\langle L, \rightarrow, ^*, \mathbf{1} \rangle$ a binary relation $\leq$ by

$$x \leq y \text{ iff } x \rightarrow y = \mathbf{1}. \tag{5}$$

Then $\leq$ is an order relation on L and $\mathbf{1}$ is the greatest element in L and, by defining two binary operations $\wedge$ and $\vee$ on a Wajsberg algebra L via

$$x \vee y = (x \rightarrow y) \rightarrow y, \tag{6}$$

$$x \wedge y = (x^* \vee y^*)^*. \tag{7}$$

we obtain a lattice $\langle L, \leq, \wedge, \vee \rangle$. Moreover, by defining on a Wajsberg algebra L two binary operation $\odot$, for each $x, y \in L$, via

$$x \odot y = (x \rightarrow y^*)^*. \tag{8}$$

$$x \oplus y = x^* \rightarrow y, \tag{9}$$

Survey of Theory and Applications of Łukasiewicz-Pavelka Fuzzy Logic

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1 Introduction

Two valued logic fits well to mathematical reasoning, indeed, it is the 'meta-mathematics of mathematics'. Applying two valued logic outside mathematics, however, rises anomalies that we cannot accept as they contradict our everyday experiences. An alternative approach to avoid paradoxes rising from seeing real world's phenomena only black or white is to accept more than two truth values *true* and *false*. This is the starting point of many-valued logics and fuzzy logic. Following Lotfi Zadeh [12], the inventor of Fuzzy Set theory, we distinguish fuzzy logic in *broad sense* i.e. everything concerning vagueness and fuzziness, from fuzzy logic in *narrow sense*, i.e. the formal logical calculus of fuzziness. In this survey we focus on the latter.

Jan Łukasiewicz [8] was the first to investigate systematically many-valued logics in 1920's. In 1935, Morchaj Wajsberg showed[1] that infinite valued sentential logic was complete with respect to the axioms conjectured by Łukasiewicz. Twenty-three years later, in 1958, C.C. Chang introduced MV-algebras, which allowed him to give another completeness proof for Łukasiewicz logic. For decades many-valued logic was far from the mainstreams of mathematical research, it was only after the 'fuzzy boom' started in 1965 with Zadeh's seminal paper 'Fuzzy Sets' that the situation has changed a bit. In 1979 Jan Pavelka [10] published a paper entitled 'On fuzzy logic' in which he generalized Łukasiewicz's logic by introducing fuzzy consequence operations, general fuzzy rules of inference, fuzzy proofs, etc. Pavelka studied the real unit interval valued fuzzy sentential logic and proved that necessary and sufficient condition for the completeness of his logic is the continuity of the implication operation. In this survey we recall an outline of Łukasiewicz-Pavelka fuzzy logic.

The traditional Greek notion of *analogia*, meaning 'proportion' is usually taken to assert the similarity, or partial identity of two objects. Following Leibniz, the founder of mathematical logic, the identity of two objects A and B means that they share all their properties, hence, objects A and B may be said to be partially identical if they share some (or most) of their properties. Immanuel Kant's Logik [7] formulated the problem on analogy in the following terms: 'Analogy concludes from partial similarity of two things to total similarity according to the principle of specification: Things of one genus, which we know to degree in much, also agree in the remainder as we know it is some of

[1] Unfortunately Wajsberg's proof was never published